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## ON LOCALIZATION AND REGULARIZATION

**Mauri Miettinen**

*Department of Theoretical Physics, Uppsala University  
P.O. Box 803, S-75108, Uppsala, Sweden*

Different regularizations are studied in localization of path integrals. We discuss the effect of the choice of regularization by evaluating the partition functions for the harmonic oscillator and the Weyl character for SU(2). In particular, we solve the Weyl shift problem that arises in path integral evaluation of the Weyl character by using the Atiyah-Patodi-Singer  $\eta$ -invariant and the Borel-Weil theory.

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\*E-mail: mauri@rhea.teorfys.uu.se

## 1 Introduction

Quantum localization is a generalization Duistermaat-Heckman theorem [1] to infinite dimensions. This theorem states that if the Hamiltonian  $H$  generates a global circle, or, more generally a torus action in the phase space  $\Gamma$  then the canonical partition function is given exactly by the saddle-point approximation around the critical points of  $H$ . Extensions to calculation of quantum mechanical partition functions using phase space path integrals have been represented in e.g.[2].

We shall first consider basic ideas of localization. Then we shall carefully regularize the pertinent functional determinants arising from the path integrals. There is an ambiguity in choosing the regularization scheme because of the spectral asymmetry of first order differential operators. Therefore, the result depends on the regularization as in the case of quantum mechanical anomalies.

Finally, we are going to apply our localization to the quantization of the simple harmonic oscillator and to the evaluation of the Weyl character of spin. We shall notice that different regularizations give different energy spectra for the harmonic oscillator. We also show that the continuum coherent state path integral yields directly the correct character for spin if we choose an appropriate regularization. In particular, we will consider the relation of character formulae to the Borel-Weil theory which constructs the irreducible representations of a Lie group as holomorphic functions. Using this theory we relate the character formulae to the equivariant index of the Dolbeault complex. The result is that the path integral yields directly the correct character without an explicit Weyl shift of the highest weight.

## 2 Localization of Phase Space Path Integrals

We are interested in exact evaluation of phase space path integrals (partition functions) of the form

$$Z(T) = \int_{L\Gamma} \mathcal{D}x \operatorname{Pf} \|\omega_{ab}(x)\| \exp \left( i \int_0^T dt [\vartheta_a \dot{x}^a - H(x)] \right). \quad (1)$$

where  $\{x^a\}$  are local coordinates in  $\Gamma$ ,  $\operatorname{Pf} \|\omega_{ab}\|$  is the Liouville measure factor,  $\vartheta_a$  the symplectic potential and  $\omega_{ab} = \partial_a \vartheta_b - \partial_b \vartheta_a$ . The integration is performed over the loop space  $L\Gamma$  consisting of the phase space loops. The integrability condition [3] requires that

$$\int_{\Sigma} \omega = 2\pi n$$

for any 2-cycle  $\Sigma$  in  $\Gamma$  so that the path integral is single valued. We introduce anticommuting variables  $\psi^a$  to write  $\text{Pf } \|\omega_{ab}\|$  as a path integral

$$Z(T) = \int \mathcal{D}x \mathcal{D}\psi \exp \left( i \int_0^T dt \left[ \vartheta_a \dot{x}^a - H(x) + \frac{1}{2} \psi^a \omega_{ab} \psi^b \right] \right). \quad (2)$$

The boundary conditions are periodic also for the fermions, since they are a realization of the differentials of the bosonic coordinates.

We interpret the path integral (2) in terms of equivariant cohomology in  $L\Gamma$ . From the bosonic part of the action we get a Hamiltonian vector field in  $L\Gamma$

$$\chi_S^a = \dot{x}^a - \omega^{ab} \partial_b H$$

whose zeroes define the Hamilton's equations. The equivariant exterior derivative in  $L\Gamma$  is

$$d_S = d + \iota_S,$$

where  $\iota_S$  denotes the contraction along the vector field  $\chi_S$ . The square of  $d_S$  is the loop space Lie derivative

$$\mathcal{L}_S = d\iota_S + \iota_S d \sim \frac{d}{dt} - \mathcal{L}_H.$$

The action  $S_B + S_F$  is supersymmetric under the infinitesimal loop space supersymmetry transformations that are parametrized by a gauge fermion  $\delta\Psi$ :

$$\begin{aligned} x^a &\rightarrow x^a + \delta\Psi d_S x^a &= x^a + \delta\Psi \psi^a, \\ \psi^a &\rightarrow \psi^a + \delta\Psi d_S \psi^a &= \psi^a + \delta\Psi \chi_S^a. \end{aligned} \quad (3)$$

This implies that the action is equivariantly closed:

$$d_S(S_B + S_F) = 0.$$

By an analogue of Fradkin-Vilkovisky theorem [4] one can show that the path integral remains intact if we modify the action by  $S \rightarrow S + d_S \Psi$  where  $\Psi$  satisfies the Lie derivative condition

$$d_S^2 \Psi = \mathcal{L}_S \Psi = 0. \quad (4)$$

In the limit  $\lambda \rightarrow 0$  the path integral

$$Z_\lambda(T) = \int \mathcal{D}x^a \mathcal{D}\psi^a \exp \left( i \int_0^T dt \left[ \vartheta_a \dot{x}^a - H(x^a) + \frac{1}{2} \psi^a \omega_{ab} \psi^b + \lambda d_S \Psi \right] \right) \quad (5)$$

reduces to (2) and  $\lambda \rightarrow \infty$  gives localization.

To construct a gauge fermion  $\Psi$  we need a metric  $g$  in the phase space. The loop space Lie derivative condition (4) is satisfied if the metric  $g$  in  $\Gamma$  is invariant under the Hamiltonian action of  $H$

$$\mathcal{L}_H g = 0, \quad (6)$$

which means that  $\chi_H$  is a Killing vector field. This is a very restrictive condition for the Hamiltonian: it must generate a global U(1)-action in  $\Gamma$ . We can choose any metric which satisfies the condition (6) and average it over the group action.

We will consider the following selections for the gauge fermion:

$$\Psi_1 = \frac{1}{2} g_{ab} \dot{x}^a \psi^b$$

gives localization to the constant modes which are points of the manifold,

$$\Psi_2 = \frac{1}{2} g_{ab} \chi_H^a \psi^b$$

to the zeroes of  $\chi_H$  which we assume to be nondegenerate and isolated and

$$\Psi_3 = \frac{1}{2} g_{ab} \chi_S^a \psi^b$$

to the classical trajectories. For simplicity we use subscripts 1,2,3 in the actions and partition functions corresponding to the gauge fermions  $\Psi_{1,2,3}$ . The actions become

$$\begin{aligned} S_1 &= \int_0^T dt \left[ \left( \vartheta_a - \frac{\lambda}{2} g_{ab} \chi_H^b \right) \dot{x}^a - H + \frac{\lambda}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{\lambda}{2} \psi^a \left( g_{ab} \partial_t + \dot{x}^c g_{bd} \Gamma_{ac}^d \right) \psi^b + \frac{1}{2} \psi^a \omega_{ab} \psi^b \right], \\ S_2 &= \int_0^T dt \left[ \vartheta_a \dot{x}^a - H + \frac{\lambda}{2} g_{ab} \chi_H^a \left( \dot{x}^b - \chi_H^b \right) + \frac{\lambda}{2} \psi^a \partial_a (g_{cb} \chi_H^c) \psi^b + \frac{1}{2} \psi^a \omega_{ab} \psi^b \right], \\ S_3 &= \int_0^T dt \left[ \vartheta_a \dot{x}^a - H + \frac{\lambda}{2} g_{ab} \chi_S^a \chi_S^b + \frac{\lambda}{2} \psi^a \partial_a (g_{cb} \chi_S^c) \psi^b + \frac{1}{2} \psi^a \omega_{ab} \psi^b \right]. \end{aligned} \quad (7)$$

To take the limit  $\lambda \rightarrow \infty$  in path integrals we make the decomposition to constant modes  $x_0^a, \psi_0^a$  and to non-constant modes  $x_t^a, \psi_t^a$  and scale the non-constant modes by  $1/\sqrt{\lambda}$

$$\begin{aligned} x^a(t) &= x_0^a + \frac{1}{\sqrt{\lambda}} x_t^a, \\ \psi^a(t) &= \psi_0^a + \frac{1}{\sqrt{\lambda}} \psi_t^a. \end{aligned} \quad (8)$$

The Jacobi determinant is unity. An expansion to a quadratic order around the constant modes and the limit  $\lambda \rightarrow \infty$  gives a Gaussian path integral

$$Z_1 = \int dx_0^a d\psi_0^a \exp \left[ -iT \left( H - \frac{1}{2} \psi_0^a \omega_{ab} \psi_0^b \right) \right] Z_{\text{fl},1}(T) \quad (9)$$

where the fluctuation path integral  $Z_{\text{fl}}(T)$  is a product of fermionic and bosonic parts:

$$\begin{aligned} Z_{\text{F},1} &= \int \prod_t d\psi_t^a \exp \left\{ -\frac{i}{2} \int_0^T dt \psi_t^a g_{ab} \partial_t \psi_t^b \right\}, \\ Z_{\text{B},1} &= \int \prod_t dx_t^a \exp \left\{ \frac{i}{2} \int_0^T dt x_t^a \left[ \mathcal{R}_{ab} \partial_t - g_{ab} \partial_t^2 \right] x_t^b \right\}. \end{aligned} \quad (10)$$

Here

$$\mathcal{R}_{ab} = R_{ab} + \tilde{\Omega}_{ab}$$

is the equivariant curvature with  $R_{ab}$  the Riemannian curvature 2-form and

$$\tilde{\Omega}_{ab} = \frac{1}{2} [\nabla_b (g_{ac} \chi_H^c) - \nabla_a (g_{bc} \chi_H^c)]$$

the momentum map [5] corresponding to  $\chi_H$ ,  $\nabla$  being the covariant derivative.  $H$ ,  $\omega$ ,  $g$  and  $\mathcal{R}$  are evaluated at the constant modes. The path integral  $Z_2$  is given by a sum over the critical points  $\{x_i\}$  of the Hamiltonian:

$$Z_2 = \sum_{x_i} \frac{\exp[-iTH]}{\text{Pf } \|\partial_a \chi_H^b\|} Z_{\text{fl},2}(T). \quad (11)$$

$Z_{\text{fl},2}$  is also a product of fermionic and bosonic parts:

$$\begin{aligned} Z_{\text{F},2}(T) &= \int \prod_t d\psi_t^a \exp \left\{ \frac{i}{2} \int_0^T dt \psi_t^a \partial_a (g_{bc} \chi_H^c) \psi_t^b \right\}, \\ Z_{\text{B},2}(T) &= \int \prod_t dx_t^a \exp \left\{ \frac{i}{2} \int_0^T dt x_t^a \partial_a (g_{bc} \chi_H^c) (\delta_d^b \partial_t - \partial_d \chi_H^b) x_t^d \right\}. \end{aligned} \quad (12)$$

Here  $g$  and  $\chi_H$  are again evaluated at the constant modes. Finally, the path integral  $Z_3$  reduces to a sum over the  $T$ -periodic classical trajectories

$$Z_3 = \sum_{x_{\text{cl}}} \frac{1}{\text{Pf } \|\delta_b^a \partial_t - \partial_b \chi_H^a\|} \exp[iS_{\text{cl}}]. \quad (13)$$

In practice, it is usually a highly non-trivial problem to find the  $T$ -periodic classical trajectories of a dynamical system [6].

### 3 Regularization of Fluctuation Path Integrals

In the following all the path integrals and determinants are evaluated over periodic configurations for both the bosonic and fermionic degrees of freedom. The primes will denote that we exclude the constant modes. In real polarization the fluctuation parts in  $Z_{1,2}$  become

$$\begin{aligned} Z_{\text{fl},1} &= \frac{1}{\sqrt{\text{Det}' \|\delta_b^a \partial_t - \mathcal{R}_b^a\|}}, \\ Z_{\text{fl},2} &= \frac{1}{\sqrt{\text{Det}' \|\delta_b^a \partial_t - \partial_b \chi_H^a\|}}. \end{aligned} \quad (14)$$

It is quite important to notice that in the reduced determinants one index is covariant and another contravariant.

In Kähler polarization the fluctuations parts are, using the additional symmetries of the metric and the Riemann curvature tensor [7],

$$\begin{aligned} Z_{\text{fl},1} &= \frac{1}{\text{Det}' \|\delta_a^b \partial_t - \mathcal{R}_a^b\|}, \\ Z_{\text{fl},2} &= \frac{1}{\text{Det}' \|\delta_a^b \partial_t - \partial_a \chi_H^b\|}. \end{aligned} \quad (15)$$

These determinants are taken over the holomorphic indices. By this we mean the following: The relevant matrices can be block diagonalized

$$A = \text{diag} (A_1, A_2, \dots, A_N)$$

with blocks

$$A_k = \begin{pmatrix} a_k^+ & 0 \\ 0 & a_k^- \end{pmatrix} \equiv \begin{pmatrix} a_k & 0 \\ 0 & -a_k \end{pmatrix}.$$

The symbols  $a_k^+$  and  $a_k^-$  denote the holomorphic and antiholomorphic eigenvalues of  $A$ , and we consider only the eigenvalues corresponding to the holomorphic indices to the determinant.

We have to choose a regularization scheme for the determinants. A standard method is to apply  $\zeta$ - and  $\eta$ -functions. The  $\zeta$ -function regularization does not directly apply to first-order operators because they have an infinite number of negative eigenvalues. To take them into account we define the  $\eta$ -function for the first-order operator  $B$  by

$$\eta_B(s) = \sum_{b_n \neq 0} \text{sign}(b_n) |b_n|^{-s} + \dim \text{Ker } B = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty dt t^{(s-1)/2} \text{Tr}[B \exp(-tB^2)].$$

Analytical continuation to  $s = 0$  gives the Atiyah-Patodi-Singer  $\eta$ -invariant [8] of  $B$  that measures the spectral asymmetry of  $B$  and specifies the phase of  $\text{Det}(B)$ . The absolute value  $|\text{Det}(B)|$  is regularized using the formula

$$|\text{Det}(B)| = +\sqrt{\text{Det}(B^2)} = +\exp\left[-\frac{1}{2}\zeta'_{B^2}(0)\right].$$

In real polarization we have to evaluate the square root of a determinant of the antisymmetric operator  $B = \partial_t - A$  where  $A$  is an antisymmetric matrix. In our case  $A$  is  $\|\mathcal{R}_a^b\|$  or  $\|\partial_a \chi_H^b\|$ . By determining the spectrum of  $B$  and applying  $\zeta$ -function regularization we get, up to an inessential numerical normalization, the result

$$\frac{1}{\sqrt{\text{Det}'(\partial_t - A)}} = \prod_{n=1}^N \left| \frac{a_n/2}{\sin(a_n T/2)} \right| = \frac{1}{T^N} \hat{A}(TA), \quad (16)$$

where we have defined the function of the matrix  $X$

$$\hat{A}(X) = \prod_n \frac{x_n/2}{\sin(x_n/2)}$$

where  $x_n$  are the skew-eigenvalues of  $X$ . The result is non-negative since the negative and positive skew-eigenvalues appear in pairs. Therefore there is no ambiguity with the spectral asymmetry.

Now we consider the determinants in Kähler polarization. It is sufficient to consider the determinant of a block. Earlier we noticed that the fluctuation path integrals reduce to the determinant of the operator  $B = i\partial_t - a$ . The functional Pfaffian in (13) is also similar to this determinant. To regularize

$$\text{Det}'(B) = \prod_{n \neq 0} \left( \frac{2\pi n}{T} - a \right)$$

properly we have to take into account that  $B$  has an infinite number of negative eigenvalues. Thus there is a problem with the spectral asymmetry.

Therefore, we have to choose a regularization prescription which has a relation to quantum mechanical anomalies. In the regularization of the determinants it is not possible to maintain all the symmetries that are present in the classical theory. For example, Elitzur et al. [9] considered the corresponding fermionic problem with antiperiodic boundary conditions. They evaluated the quantum mechanical partition function for a Dirac fermion in an external gauge field  $A(t)$  in  $0 + 1$ -dimensions

$$Z(T) = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ i \int_0^T dt \bar{\psi} (i\partial_t - a) \psi \right] = \text{Det} (i\partial_t - a). \quad (17)$$

where, because of the gauge invariance of the action only the constant mode  $a$  of  $A(t)$  contributes. The classical action has both the invariance under large gauge transformations

$$\begin{aligned} a &\rightarrow a + n2\pi/T \\ \psi &\rightarrow \psi \end{aligned} \quad (18)$$

and the charge conjugation invariance

$$\begin{aligned} a &\leftrightarrow -a \\ \psi &\leftrightarrow -\bar{\psi}. \end{aligned} \quad (19)$$

However, when regularizing the determinant one has to choose which symmetry one wants to maintain, which leads to a global anomaly. Here we have an analogous situation. It is not *a priori* clear what the result of the regularization should be, and there is a genuine ambiguity.

Since the zeroes of the determinant are at  $aT = 2\pi n$ , the determinant must be proportional to

$$\frac{\sin(aT/2)}{a/2}.$$

The proportionality factor can be any function without zeroes that is, the exponent function. The determinant is therefore, up to an irrelevant constant,

$$\text{Det } (i\partial_t - a) = \frac{\sin(aT/2)}{a/2} \exp(i\phi aT)$$

with a phase  $\phi$  whose natural values turn out to be 0 and  $\pm 1/2$  since they yield the (anti)symmetries of the product under  $a \leftrightarrow -a$  and  $a \rightarrow a + 2\pi n/T$ . However, there is a minor subtlety: in our localization formulae the zero modes are absent and this destroys these symmetries. Nevertheless, we may still consider the residual symmetries. The choice  $\phi = 0$  corresponds to neglecting the spectral asymmetry and choosing the (anti)symmetry  $a \rightarrow -a$  to be unbroken. In this regularization scheme the inverse determinant is simply

$$\frac{1}{\text{Det}'(\partial_t - A)} = \frac{1}{T^N} \hat{A}(TA)$$

This is the result that usually appears in literature. However, there is another possibility. The values  $\phi = \pm 1/2$  correspond to maintaining the symmetry  $a \rightarrow a + 2\pi n/T$  and taking into account the spectral asymmetry by the Atiyah-Patodi-Singer  $\eta$ -invariant. This yields

$$\frac{1}{\text{Det}'(\partial_t - A)} = \prod_{n=1}^N \frac{a_n/2}{\sin(a_n T/2)} \exp(ia_n T/2) = \frac{1}{T^N} \text{Td}(TA) \quad (20)$$

where we have defined the following function of the matrix  $X$

$$\text{Td}(X) = \prod_n \frac{x_n/2}{\sin(x_n/2)} e^{ix_n/2}.$$

We take only the eigenvalues corresponding to the holomorphic indices to the determinant.

Let us now write down the resulting localization formulae. The localization to constant modes yields the expression

$$Z_1(T) = \frac{1}{T^N} \int dx_0^a d\psi_0^a \text{Ch}[-iT(H - \omega)] \begin{Bmatrix} \hat{A}(T\mathcal{R}) \\ \text{Td}(T\mathcal{R}) \end{Bmatrix}. \quad (21)$$

We have defined equivariant generalizations [10] of the conventional characteristic classes known as equivariant  $\hat{A}$ - and Todd-genus, and identified the exponential with the equivariant Chern class. When  $H = 0$  they reduce to the conventional characteristic classes

and the result is a topological invariant. The localization to the critical points  $\{x_i\}$  of the Hamiltonian gives the result

$$Z_2(T) = \frac{1}{T^N} \sum_{x_i} \frac{\exp(-iTH)}{\text{Pf}(\partial\chi_H)} \begin{Bmatrix} \hat{A}(T\partial\chi_H) \\ \text{Td}(T\partial\chi_H) \end{Bmatrix}. \quad (22)$$

We must use local coordinates in the evaluation of the determinants when localizing to the critical points of the Hamiltonian. Finally, the localization to  $T$ -periodic classical trajectories yields

$$Z_3(T) = \frac{1}{T^N} \sum_{x_{\text{cl}}} \exp(iS_{\text{cl}}) \begin{Bmatrix} \hat{A}(T\partial\chi_H) \\ \text{Td}(T\partial\chi_H) \end{Bmatrix}. \quad (23)$$

## 4 Harmonic Oscillator

Now we show that the localization formulae yield the correct partition function for the harmonic oscillator in a flat phase space. The path integral for it is Gaussian and in principle there is no reason to apply localization to it. However, it is reasonable to check by some simple examples that our assumptions and derivations are valid. In particular, we will show that the choice of the metric in the phase space is not relevant, contrary to claims in literature [11]. It is also illustrative to consider the significance of the regularization schemes we have used.

In real polarization the Hamiltonian is  $H = \frac{1}{2}(p^2 + q^2)$  and the symplectic 2-form is  $dq \wedge dp$ . The coherent state representation (Kähler polarization) requires some further investigation, since we have to fix an operator ordering prescription. In terms of creation and annihilation operators the normal and symmetric ordered Hamiltonians are, respectively,

$$\begin{aligned} H_n &= : \frac{1}{2} (a^\dagger a + a a^\dagger) := a^\dagger a, \\ H_s &= a^\dagger a + \frac{1}{2}. \end{aligned} \quad (24)$$

The symmetric ordered Hamiltonian has an explicit zero point energy  $E_0 = 1/2$ .

To apply the localization formulae we must choose a metric in the phase space and calculate the equivariant curvature and the derivatives of the Hamiltonian vector field. If the Lie-derivative condition  $\mathcal{L}_H g = 0$  is satisfied we can start from an any smooth metric in the phase space and average it. So we may choose a constant metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The non-zero components of the equivariant curvature are

$$\mathcal{R}_q^p = -\mathcal{R}_p^q = 1.$$

The localization formula (21) yields the result

$$\begin{aligned} Z(T) &= \int dp dq d\psi^q d\psi^p \exp \left[ -iT \left( \frac{1}{2}p^2 + \frac{1}{2}q^2 - \psi^p \psi^q \right) \right] \frac{1}{2 \sin(T/2)} \\ &\sim \frac{1}{2 \sin(T/2)} = \sum_{n=0}^{\infty} \exp[i(n+1/2)T] \end{aligned} \quad (25)$$

which is the correct partition function with the zero-point energy  $E_0 = 1/2$ .

Let us now digress slightly to discuss the result. In [11] Dykstra, Lykken and Reiten analyzed this problem and they noticed a dependence on the metric. What they did not notice was that the index structure of the equivariant curvature is  $\mathcal{R}_a^b$  and therefore it is invariant under global scalings of the metric. Furthermore, they used a metric which in polar coordinates near the origin behaves like

$$ds^2 = dr^2 + cr^2 d\phi^2.$$

This is a metric on a cone, not on a plane when  $c \neq 1$  and is not smooth, nor even continuous at the origin. Therefore it is not surprising that their energy levels depend on the parameter  $c$  which represents the tip angle of the cone. From this we indeed see that we cannot choose an arbitrary invariant metric, since it has to respect the topology of the phase space.

The localization to the critical points of the Hamiltonian (22) yields also the correct result. The only zero of  $\chi_H$  is the origin of the phase space, which gives

$$Z(T) = \frac{1}{\text{Pf} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \frac{1/2}{\sin(T/2)} = \frac{1}{2 \sin(T/2)}.$$

If we want to apply the localization the classical trajectories we must classify all the  $T$ -periodic classical trajectories. If  $T \neq 2\pi n$  the problem reduces to the localization to the critical points of the Hamiltonian. However, if  $T = 2\pi n$  the zeroes of  $\chi_S$  are not isolated and we have to use a degenerate version of the localization formula to the classical trajectories [12]

We now consider the harmonic oscillator in the Kähler polarization. We will only discuss the localization formulae to constant modes. The reasoning is similar with other formulae. There are four cases to consider: the localizations with the  $\hat{A}$ -genus and Todd-genus using two different orderings. We will only list the spectra we obtain. The use of  $\hat{A}$ -genus yields the spectra  $E_n = n + 1/2$  (normal ordering) and  $E_n = n + 1$  (symmetric ordering). The Todd-genus gives the results  $E_n = n$  (normal ordering) and  $E_n = n + 1/2$  (symmetric ordering). The first and fourth results have the correct zero-point energy. From this example we see that to get correct results from the path integral we do need some additional information other than the classical action and boundary conditions: we must choose a regularization scheme that gives physically correct results.

## 5 Character for $SU(2)$

We shall now use our localization formulae to derive the Kirillov and Weyl character formulae for Lie groups [10]. The character formula for  $SU(2)$  has been widely discussed in literature [13, 14, 15]. However, there has been some controversy about the Weyl shift problem: the path integral usually gives almost the correct character up to the substitution  $j \rightarrow j + 1/2$ . We show that the coherent state path integral and the localization formulae with the Todd-genus directly yield the correct character. In this calculation we use the continuum version of the coherent state path integral and show that this also yields the correct result, contrary to discussions in literature [16].

To motivate the use of Todd-genus we relate the character of a simple Lie group  $G$  in the highest weight representation  $\lambda$  to the index of the twisted Dolbeault complex on the coadjoint orbit  $O_f$  [17] of the group. The Borel-Weil theory [14, 18] constructs the irreducible representations of  $G$  as holomorphic sections of a line bundle  $L$  that is associated to a principal bundle  $G \rightarrow G/T_G \sim O_f$  where  $T_G$  is the Cartan torus of  $G$  [18]. The holomorphic sections of this line bundle (coherent states) form the basis for the irreducible representation. The connection 1-form on  $L$  is the symplectic potential

$$\vartheta = \frac{\partial F}{\partial z^k} dz^k - \frac{\partial F}{\partial \bar{z}^k} d\bar{z}^k$$

where  $F$  is the Kähler potential on  $O_f$ . It can be shown that the twisted Dolbeault operator  $\bar{\partial}_L = \bar{\partial} + \vartheta_{\bar{z}}$  annihilates the normalized coherent states  $|z\rangle$  and therefore  $|z\rangle \in H^{0,0}(O_f, L)$ . If we can prove that all the other cohomology groups are trivial, e.g. by Lichnerowicz vanishing theorem [10], we conclude that the dimension of the highest weight representation  $R_\lambda$  is  $\dim H^{0,0}(O_f, L)$ . Consequently, this is equal to the index of the twisted Dolbeault complex. The Riemann-Roch-Hirzebruch index theorem relates this analytical index to the topological invariant

$$\text{ind } \bar{\partial}_L = \dim R_\lambda = \int_{O_f} \text{Td}(O_f) \wedge \text{Ch}(L) . \quad (26)$$

Indeed, we notice that the localization formula (21) with  $H = 0$  represents this index provided we use the Todd-class. For  $SU(2)$  we obtain the known result for the dimension of the spin- $j$ - representation

$$\dim R_j = \text{ind } \bar{\partial}_L = 2j + 1 ,$$

This is the correct result without the explicit Weyl shift by the Weyl vector  $\rho = 1/2$ .

We shall now use an equivariant version of the index theorem to derive the character formulae. The character of an element in the Cartan subalgebra is the partition function for the Hamiltonian  $H$  that represents it on  $O_f$ :

$$\chi(\beta) = \text{Str } \exp[-iTH] . \quad (27)$$

To make a relation to the Dolbeault index we write this as an equivariant index (character index, G-index, Lefschetz number) [10]. One can show that the Laplacians  $\bar{\partial}_L^\dagger \bar{\partial}_L$  and  $\bar{\partial}_L \bar{\partial}_L^\dagger$  have equal non-zero eigenvalues. If all other comohomology classes except  $H^{0,0}$  are trivial, as we presume,  $\bar{\partial}_L^\dagger$  does not have zero modes. Consequently, we can write the trace as an equivariant index:

$$\begin{aligned} \text{ind }_H(\bar{\partial}_L, T) &\equiv \lim_{\beta \rightarrow \infty} \text{Tr } e^{-iT H} (e^{-\beta \bar{\partial}_L^\dagger \bar{\partial}_L} - e^{-\beta \bar{\partial}_L \bar{\partial}_L^\dagger}) \\ &= \lim_{\beta \rightarrow \infty} \text{Str } \exp[-iT H] \exp \left[ -\beta \begin{pmatrix} \bar{\partial}_L^\dagger \bar{\partial}_L & 0 \\ 0 & \bar{\partial}_L \bar{\partial}_L^\dagger \end{pmatrix} \right] \end{aligned} \quad (28)$$

Only the zero modes contribute to the trace. The expression is also independent of  $\beta$ . Therefore, in the limit  $\beta \rightarrow 0$ , all we are left with are the zero modes of  $\bar{\partial}_L$ . Consequently, the equivariant index is equal to the character

$$\text{ind }_H(\bar{\partial}_L, T) = \text{Str } \exp[-iT H]$$

Thus, the character is the equivariant index of the twisted Dolbeault complex and therefore we choose the localization with the Todd-class.

To derive the character formulae we apply standard methods to write  $\text{Str } \exp[-iT H]$  as a coherent state path integral of the form (2). Since we can choose an invariant metric on a coadjoint orbit [17] we can localize the path integral to classical trajectories, to constant modes or to critical points of the Hamiltonian. The two latter cases yield the Kirillov character formula [19] ( $2N$  is the dimension of the orbit)

$$\chi(T) = \frac{1}{T^N} \int_{O_f} \text{Ch}[-iT(H - \omega)] \text{Td}(T\mathcal{R}^+) , \quad (29)$$

and the Weyl character formula

$$\chi(T) = \frac{1}{T^N} \sum_{z_i} \frac{\exp(-iT H)}{\det^+(\partial \chi_H)} \text{Td}(T\partial \chi_H) , \quad (30)$$

respectively. In (30) we have identified the Pfaffian in the real polarization with the determinant over the holomorphic eigenvalues of  $\|\partial_a \chi_H^b\|$  and the summation is over the critical points of the Hamiltonian or equivalently the Weyl group.

As the only example we evaluate the character for  $SU(2)$ . We write the character as a coherent state path integral over the coadjoint orbit  $SU(2)/U(1) \sim S^2$ . We choose complex coordinates by introducing the stereographic projection from the south pole. The Kähler potential on the orbit with radius  $j$  is  $F = j \log(1 + z\bar{z})$  from which we obtain the metric and the symplectic 1- and 2-forms in the standard fashion. The integrability condition requires  $j$  to be a multiple of  $1/2$ : this is the topological quantization of spin. The canonical realization for  $H = J_3$  is

$$J_3 = -j \frac{1 - z\bar{z}}{1 + z\bar{z}}$$

and the path integral for the character becomes (2)

$$\chi_j(T) = \int \mathcal{D}z \mathcal{D}\bar{z} \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ ij \int_0^T dt \left( i \frac{\dot{z}\bar{z} - z\dot{\bar{z}}}{1+z\bar{z}} + \frac{1-z\bar{z}}{1+z\bar{z}} + \frac{2i\psi\bar{\psi}}{(1+z\bar{z})^2} \right) \right] \quad (31)$$

with periodic boundary conditions. The Lie-derivative condition (6) is satisfied for  $H = J_3$ . This path integral is given exactly by the WKB-approximation [2, 16]. The relevant quantities in the Kirillov formula (29) are

$$\begin{aligned} H - \omega &= j \frac{1 - z\bar{z} - \psi\bar{\psi}}{1 + z\bar{z} + \psi\bar{\psi}}, \\ \mathcal{R}^+ &= R^+ + \Omega^+ = \frac{1 - z\bar{z} - \psi\bar{\psi}}{1 + z\bar{z} + \psi\bar{\psi}}. \end{aligned} \quad (32)$$

Now one can use the Parisi-Sourlas integration formula

$$\int dz d\bar{z} d\psi d\bar{\psi} F(z\bar{z} + \psi\bar{\psi}) = \pi[F(\infty) - F(0)]$$

which gives

$$\chi_j(T) = \frac{\sin(j + 1/2)T}{\sin(T/2)} = \sum_{m=-j}^j \exp[imT]. \quad (33)$$

This is exactly the correct result without an explicit Weyl shift. Also the Weyl formula (30) gives the correct result when we use local coordinate charts in the vicinity of the critical points. To get the correct north pole contribution we invert the coordinates  $z \rightarrow 1/z, \bar{z} \rightarrow 1/\bar{z}$ . This also yields the correct character (33):

$$\chi_j(T) = \frac{\exp[-ijT]}{2\sin(T/2)} \exp[-iT/2] + \frac{\exp[-ijT(-1)]}{2\sin(-T/2)} \exp[iT/2] = \frac{\sin(j + 1/2)T}{\sin(T/2)} = \sum_{m=-j}^j \exp[imT].$$

On the other hand, using  $\hat{A}$ -genus we obtain the result

$$\chi_j(T) = \frac{\sin(jT)}{\sin(T/2)}$$

which is the correct result up to the Weyl shift  $j \rightarrow j + 1/2$ . So we see that in the character formulae we have to use the Todd-genus instead of  $\hat{A}$ -genus to directly get the correct result.

## 6 Conclusions

We have considered phase space path integrals with the property that the Hamiltonian generates an isometry of the phase space. Using equivariant cohomology in the loop space

we were able to reduce the path integrals to finite dimensional integrals and sums. We also noticed that the results were not uniquely defined because of spectral asymmetry. The choice of regularization yielded equivariant  $\hat{A}$ - and Todd-classes.

We applied localization to the harmonic oscillator and to the quantization of coadjoint orbits. We showed that localization produces correct results for these systems. In addition, we derived Kirillov and Weyl character formulae that produce correct characters for Lie groups without the Weyl shift. We demonstrated this explicitly by evaluating the character for  $SU(2)$ . The explanation for the Weyl shift was the same as in the case of the Coxeter shift [20] in Chern-Simons theory, the  $\eta$ -invariant.

It would be interesting to apply our formalism to more complicated systems such as loop groups and field theories. Also, it seems possible to use localization and equivariant cohomology to study quantum integrability, generic supersymmetric theories and problems in classical mechanics, as well.

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